Note: screenshots for row reducing a matrix with technology are on last page

§2.2 cont'd

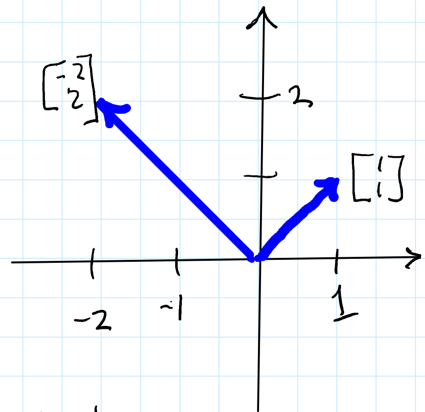
recall given vectors $\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\2\end{bmatrix}$

$F = \begin{bmatrix}1 & -1\\1 & 1\end{bmatrix}$

get resulting vectors:

$\begin{bmatrix}1 & -1\\1 & 1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1+0\\1+0\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix}$

$\begin{bmatrix}1 & -1\\1 & 1\end{bmatrix}\begin{bmatrix}0\\2\end{bmatrix} = \begin{bmatrix}0-2\\0+2\end{bmatrix} = \begin{bmatrix}-2\\2\end{bmatrix}$

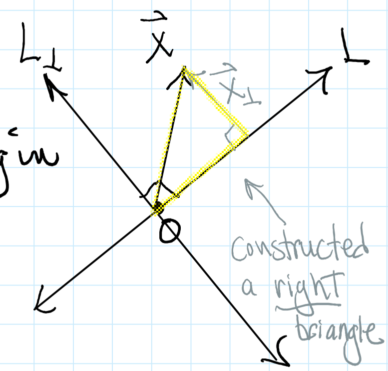


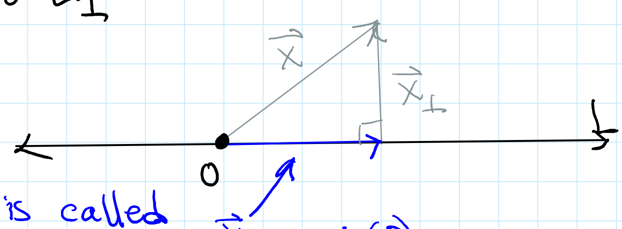
Generalize: $\begin{bmatrix}k & 0\\0 & k\end{bmatrix}$ if $0<k\leq 1$ dilates | contracts shape by a scalar of k

Orthogonal Projection

consider line L on the plane] running through origin
vector $\vec{x}$ "

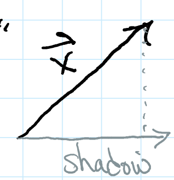
next construct $L_\perp$ through origin
$\vec{x}_\perp$ from L to $\vec{x}$
s.t. $\vec{x}_\perp \parallel$ to $L_\perp$





3rd side of constructed right triangle is called
orthogonal projection of $\vec{x}$ onto L
denoted as $proj_L(\vec{x}) = \vec{x}_\parallel$ $\parallel$ because is a scalar of L

think of as a "shadow" cast by $\vec{x}$



def'n: length (or norm) of $\vec{x}$ in $\mathbb{R}^n$

shadow

def'n: length (or <u>norm</u>) of $\vec{x}$ in $\mathbb{R}^n$

$$\|\vec{x}\| = \sqrt{\vec{x}\cdot\vec{x}}$$

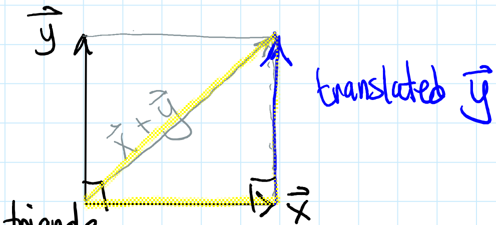
↑ dot product

$$= \sqrt{(x_1)^2 + (x_2)^2 + \dots + (x_n)^2}$$

ex. find $\|\vec{x}\|$ given $\vec{x} = \begin{bmatrix} 7 \\ 1 \\ 7 \\ -1 \end{bmatrix}$ $\quad \|\vec{x}\| = \sqrt{49+1+49+1} = \sqrt{100} = \boxed{10}$

special case: vector $\vec{u}$ is called the <u>unit vector</u> if $\|\vec{u}\| = 1$

<u>Dot Product of Orthogonal Vectors</u>



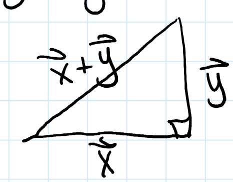
given 2 vectors $\vec{x}, \vec{y}$ in $\mathbb{R}^2$ $\quad \vec{x} \perp \vec{y}$

then $\vec{x} + \vec{y}$ creates a parallelogram

after translating $\vec{y}$, once again created right triangle

then, by Pythagorean Theorem, $\quad \|\vec{x}\|^2 + \|\vec{y}\|^2 = \|\vec{x}+\vec{y}\|^2$

by def'n of norm: $\sqrt{\vec{x}\cdot\vec{x}}^{\,2}$



$$\vec{x}\cdot\vec{x} + \vec{y}\cdot\vec{y} = (\vec{x}+\vec{y})\cdot(\vec{x}+\vec{y})$$

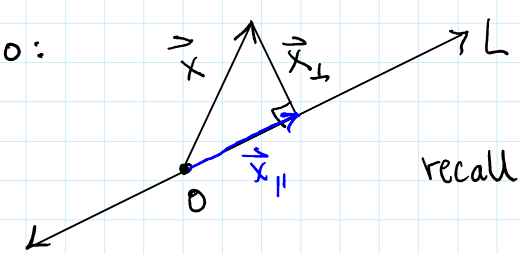
FOIL

$$\cancel{\vec{x}\cdot\vec{x}} + \cancel{\vec{y}\cdot\vec{y}} = \cancel{\vec{x}\cdot\vec{x}} + 2(\vec{x}\cdot\vec{y}) + \cancel{\vec{y}\cdot\vec{y}}$$

$$0 = 2(\vec{x}\cdot\vec{y})$$

conclusion: $\boxed{\vec{x}, \vec{y} \text{ in } \mathbb{R}^n \text{ are orthogonal if } \vec{x}\cdot\vec{y} = 0}$ $\quad 0 = \vec{x}\cdot\vec{y}$

back to previous scenario:



$$\vec{x} = \vec{x}_\parallel + \vec{x}_\perp$$

recall $\vec{x}_\parallel = \text{proj}_L(\vec{x})$

let $\vec{w}$ = non-zero vector $\parallel$ to L

then can express $\text{proj}_L(\vec{x}) = k\vec{w}$

$$\vec{x} = k\vec{w} + \vec{x}_\perp$$

$$\vec{x}_\perp = \underbrace{\vec{x} - k\vec{w}}_{\perp \text{ to } L}$$

dot product is zero: $\quad \vec{x}_\perp \cdot \vec{w} = 0$

$(\vec{x} - k\vec{w})\cdot\vec{w} = 0$

$\perp$ to L   dot product is zero : $\vec{x}_\perp \cdot \vec{w} = 0$

$$(\vec{x} - k\vec{w}) \cdot \vec{w} = 0$$
$$\vec{x}\cdot\vec{w} - k(\vec{w}\cdot\vec{w}) = 0$$
$$\vec{x}\cdot\vec{w} = k(\vec{w}\cdot\vec{w})$$
$$\frac{\vec{x}\cdot\vec{w}}{\vec{w}\cdot\vec{w}} = k$$

$\therefore \text{proj}_L(\vec{x}) = k\vec{w}$

$$\text{proj}_L(\vec{x}) = \left(\frac{\vec{x}\cdot\vec{w}}{\vec{w}\cdot\vec{w}}\right)\vec{w}$$

SPECIAL CASE:

let $\vec{w} = \vec{u}$ (unit vector)

then $\text{proj}_L(\vec{x}) = \left(\frac{\vec{x}\cdot\vec{u}}{\cancel{\vec{u}\cdot\vec{u}}}\right)\vec{u} = (\vec{x}\cdot\vec{u})\vec{u}$ $\boxed{\text{proj}_L\vec{x} = (\vec{x}\cdot\vec{u})\vec{u}}$

since $\|\vec{u}\| = 1$

next, develop the corresponding transformation matrix given $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$

then
$$\text{proj}_L(\vec{x}) = (\vec{x}\cdot\vec{u})\vec{u}$$
$$= \left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\cdot\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right)\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$
$$= (x_1u_1 + x_2u_2)\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

treat like a scalar : $k\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} ka \\ kb \end{bmatrix}$

$$= \begin{bmatrix} (x_1u_1 + x_2u_2)u_1 \\ (x_1u_1 + x_2u_2)u_2 \end{bmatrix}$$
$$= \begin{bmatrix} x_1(u_1)^2 + x_2u_1u_2 \\ x_1u_1u_2 + x_2(u_2)^2 \end{bmatrix}$$

$$\text{proj}_L(\vec{x}) = \begin{bmatrix} (u_1)^2 & u_1u_2 \\ u_1u_2 & (u_2)^2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$T(\vec{x}) = [\quad * \quad]\,\vec{x}$$
matrix P

generalize: if $\vec{w}$ is non-zero vector $\parallel$ to L

then $\vec{u}$ is the unit vector in the direction of $\vec{w}$

$\vec{u} = \frac{\vec{w}}{\|\vec{w}\|}$ then $u_1 = \frac{w_1}{\sqrt{(w_1)^2 + (w_2)^2}}$ $u_2 = \frac{w_2}{\sqrt{(w_1)^2 + (w_2)^2}}$

$$P = \begin{bmatrix} (u_1)^2 & u_1u_2 \\ \end{bmatrix} = \begin{bmatrix} \frac{(w_1)^2}{(w_1)^2 + (w_2)^2} & \frac{w_1w_2}{(w_1)^2 + (w_2)^2} \\ \end{bmatrix}$$

$$P = \begin{bmatrix} (u_1)^2 & u_1 u_2 \\ u_1 u_2 & (u_2)^2 \end{bmatrix} = \begin{bmatrix} \dfrac{(\omega_1)^2}{(\omega_1)^2 + (\omega_2)^2} & \dfrac{\omega_1 \omega_2}{(\omega_1)^2 + (\omega_2)^2} \\ \dfrac{\omega_1 \omega_2}{(\omega_1)^2 + (\omega_2)^2} & \dfrac{(\omega_2)^2}{(\omega_1)^2 + (\omega_2)^2} \end{bmatrix}$$

*re-written from above

$$P = \frac{1}{(\omega_1)^2 + (\omega_2)^2} \begin{bmatrix} (\omega_1)^2 & \omega_1 \omega_2 \\ \omega_1 \omega_2 & (\omega_2)^2 \end{bmatrix}$$

Row Reduce using Maple:

$with(LinearAlgebra):$
$A := Matrix([[1,3,4],[2,4,2],[3,7,6]]);$

$$A := \begin{bmatrix} 1 & 3 & 4 \\ 2 & 4 & 2 \\ 3 & 7 & 6 \end{bmatrix}$$

$ReducedRowEchelonForm(A);$

$$\begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

Can also row reduce using TI-84: